Lec 2: Theoretical Tools

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April 12, 2026

Objectives

1. Unconstrained optimization
2. Constrained optimization
3. Applications
4. Equilibrium and Social Welfare

THEORETICAL AND EMPIRICAL TOOLS

Theoretical tools: The set of tools designed to understand the mechanics behind economic decision making.

Economists model individuals' choices using the concepts of utility function maximization subject to budget constraint

Empirical tools: The set of tools designed to analyze data and answer questions raised by theoretical analysis.

- We will do the first in this chapter and the empirical one in the following chapter

Unconstrained optimization

- Find the minimum of the following marginal cost function:

  $MC = q^2 - 4q + 10$ where q is quantity produced

UTILITY MAPPING OF PREFERENCES

Utility function: A utility function is some mathematical function translating consumption into utility:

$$U = u(X_1, X_2, X_3, ...)$$

where X_1, X_2, X_3, and so on are the quantity of goods 1,2,3,... consumed by the individual

Example with two goods: $u(X_1, X_2) = \sqrt{X_1 \cdot X_2}$ with X_1 number of movies, X_2 number of music songs

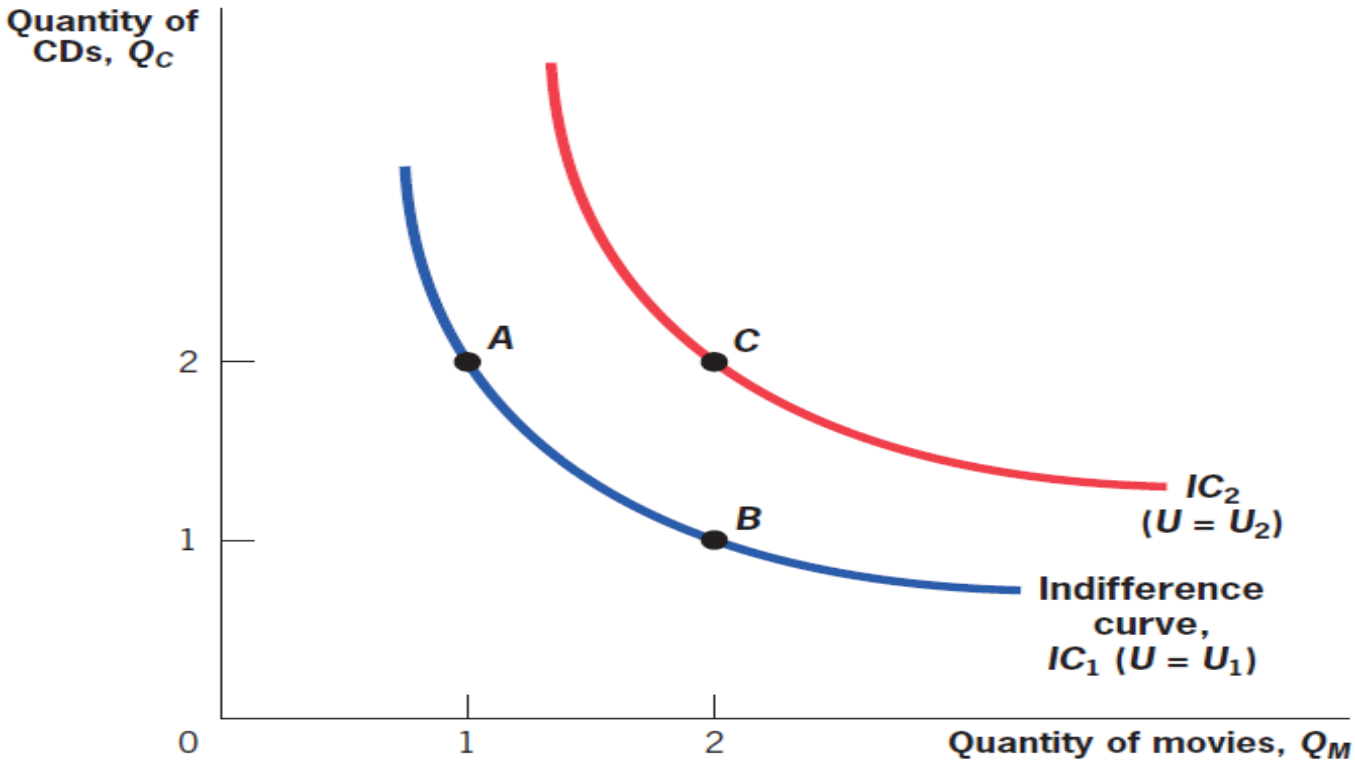
Individual utility increases with the level of consumption of each good

PREFERENCES AND INDIFFERENCE CURVES

- **Indifference curve (IC):** A graphical representation of all bundles of goods that make an individual equally well off; ICs are the level curves of the utility function
- Mathematically, indifference curve giving utility level $\bar{U}$ is given by the set of bundles (X_1, X_2) such that $u(X_1, X_2) = \bar{U}$
- Indifference curves have two essential properties, both of which follow naturally from the more-is-better assumption:
  1. ICs are always downward sloping.
  2. Higher ndifference curves (further from the origin) imply higher utility and hence are preferred.
  3. ICs shouldn't intersect

2.1

Preferences and Indifference Curves



MARGINAL UTILITY

Marginal utility: The additional increment to utility obtained by consuming an additional unit of a good:

Marginal utility of good 1 is defined as:

$$MU_1 = \frac{\partial u}{\partial X_1} \simeq \frac{u(X_1 + dX_1, X_2) - u(X_1, X_2)}{dX_1}$$

It is the derivative of utility with respect to X_1 keeping X_2 constant (called the partial derivative)

Example:

$$u(X_1, X_2) = \sqrt{X_1 \cdot X_2} \Rightarrow \frac{\partial u}{\partial X_1} = \frac{\sqrt{X_2}}{2\sqrt{X_1}}$$

This utility function described exhibits the important principle of **diminishing marginal utility**: $\partial u/\partial X_1$ decreases with X_1: the consumption of each additional unit of a good gives less extra utility than the consumption of the previous unit

MARGINAL RATE OF SUBSTITUTION

Marginal rate of substitution (MRS): The *MRS* is equal to (minus) the slope of the indifference curve, the rate at which the consumer will trade the good on the vertical axis for the good on the horizontal axis.

Marginal rate of substitution between good 1 and good 2 is:

$$MRS_{1,2} = \frac{MU_1}{MU_2}$$

Individual is indifferent between 1 unit of good 1 and $MRS_{1,2}$ units of good 2.

Example:

$$u(X_1, X_2) = \sqrt{X_1 \cdot X_2} \Rightarrow MRS_{1,2} = \frac{X_2}{X_1}$$

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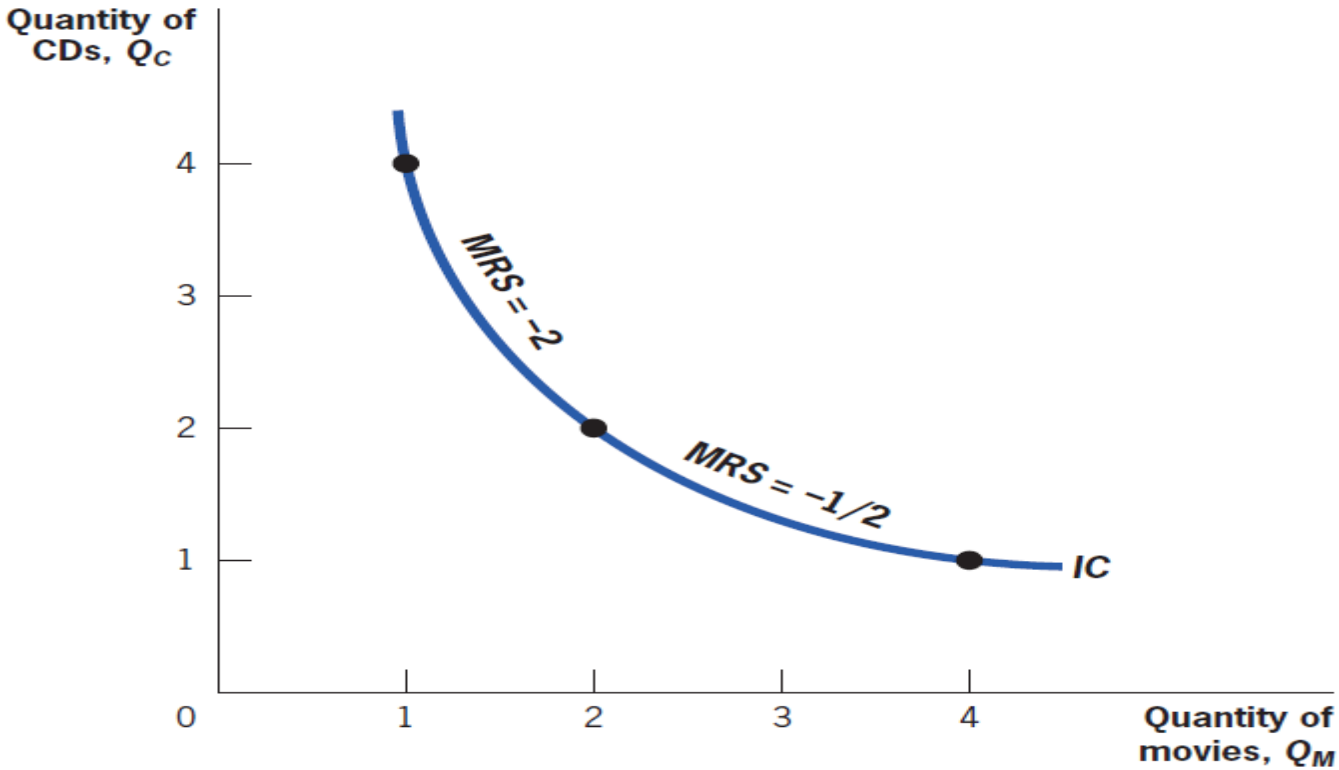
Marginal Rate of Substitution



BUDGET CONSTRAINT

Budget constraint: A mathematical representation of all the combinations of goods an individual can afford to buy if she spends her entire income.

$$p_1 X_1 + p_2 X_2 = Y$$

with p_i price of good i, and Y disposable income.

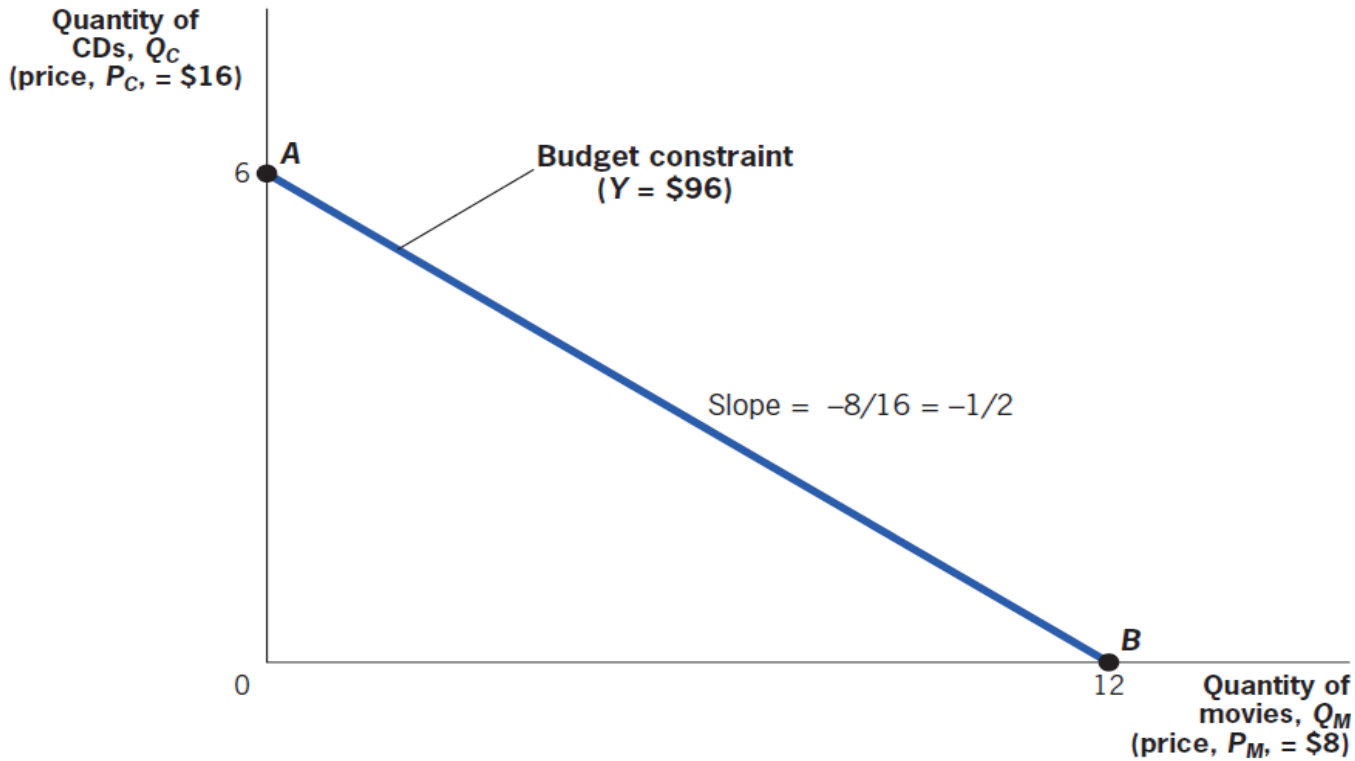
Budget constraint defines a linear set of bundles the consumer can purchase with its disposable income Y

$$X_2 = \frac{Y}{p_2} - \frac{p_1}{p_2} X_1$$

The slope of the budget constraint is $-p_1/p_2$

2.1

Budget Constraints



UTILITY MAXIMIZATION

Individual maximizes utility subject to budget constraint:

$$\max_{X_1,X_2} u(X_1, X_2) \quad \text{subject to} \quad p_1X_1 + p_2X_2 = Y$$

Solution graphically:

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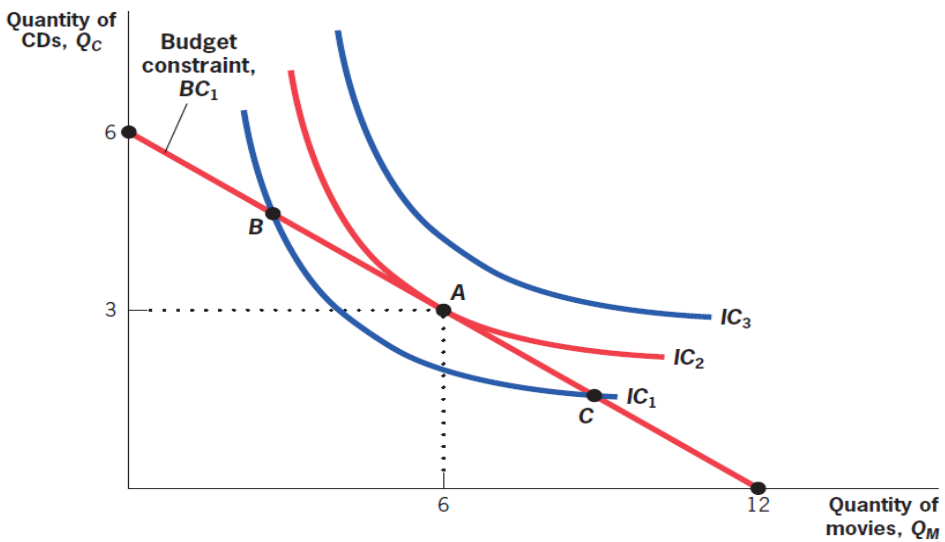
Putting It All Together: Constrained Choice



The solution Mathematically:

- $MRS_{1,2} = \frac{p_1}{p_2}$ ...(1)
- $p_1 X_1 + p_2 X_2 = Y$ ...(2)

Example:

$$\max_{x_1, x_2} u = x_1 \cdot x_2 \quad \text{subject to} \quad x_1 + 2x_2 = 20$$

Demand Curve: A curve showing the quantity of a good demanded by individuals at each price.

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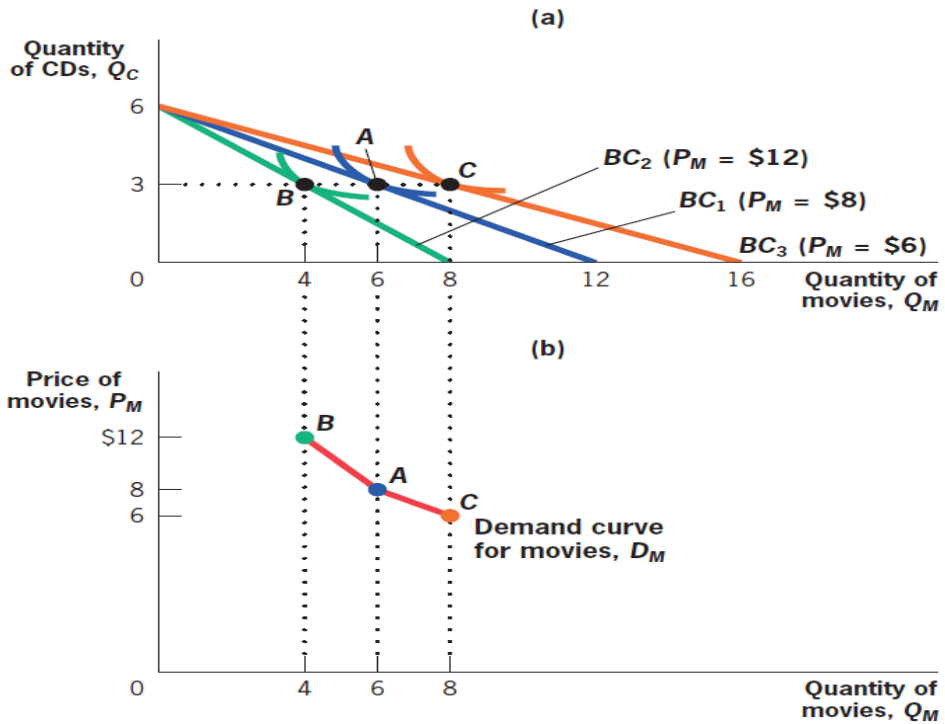
Demand Curves



INCOME AND SUBSTITUTION EFFECTS

Let us denote by $p = (p_1, p_2)$ the price vector

Individual maximization generates demand functions $X_1(p, Y)$ and $X_2(p, Y)$

How does $X_1(p, Y)$ vary with p and Y?

Those are called price and income effects

Example: $u(X_1, X_2) = \sqrt{X_1 \cdot X_2}$ then $MRS_{1,2} = X_2/X_1$.

Utility maximization implies $X_2/X_1 = p_1/p_2$ and hence $p_1X_1 = p_2X_2$

Budget constraint $p_1X_1 + p_2X_2 = Y$ implies $p_1X_1 = p_2X_2 = Y/2$

Demand functions: $X_1(p, Y) = Y/(2p_1)$ and $X_2(p, Y) = Y/(2p_2)$

INCOME EFFECTS

Income effect is the effect of giving extra income Y on the demand for goods: How does $X_1(p, Y)$ vary with Y?

Normal goods: Goods for which demand increases as income Y rises: $X_1(p, Y)$ increases with Y (most goods are normal)

Inferior goods: Goods for which demand falls as income Y rises: $X_1(p, Y)$ decreases with Y (example: you use public transportation less when you are rich enough to buy a car)

Example: if leisure is a normal good, you work less (i.e. get more leisure) if you are given a transfer

PRICE EFFECTS (Total Effect of Price Change)

How does $X_1(p_1, p_2, Y)$ vary with p_1?

Changing p_1 affects the slope of the budget constraint and can be decomposed into 2 effects:

1. **Substitution effect**: Holding utility constant, a relative rise in the price of a good will always cause an individual to choose less of that good
2. **Income effect**: A rise in the price of a good will typically cause an individual to choose less of all goods because her income can purchase less than before

For normal goods, an increase in p_1 reduces $X_1(p_1, p_2, Y)$ through both substitution and income effects

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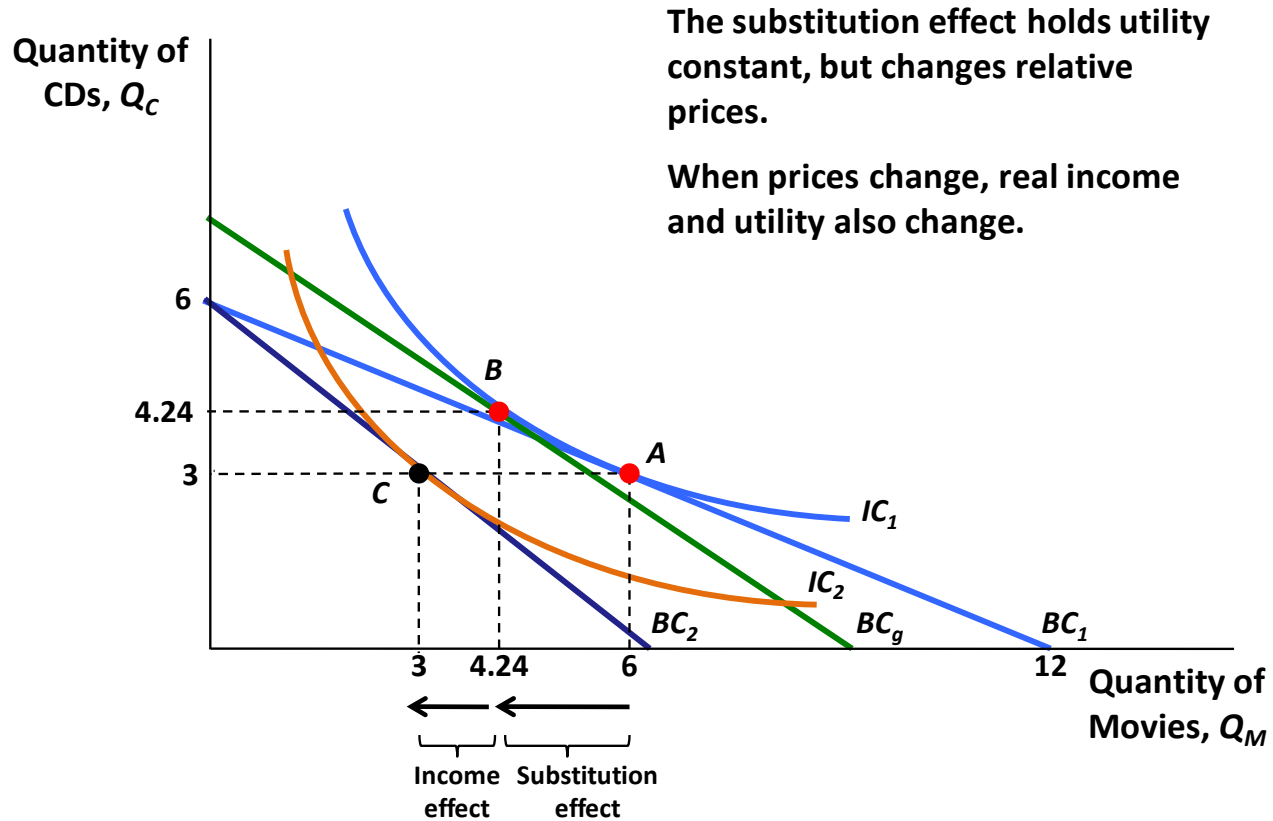
The Effects of Price Changes: Substitution and Income Effects



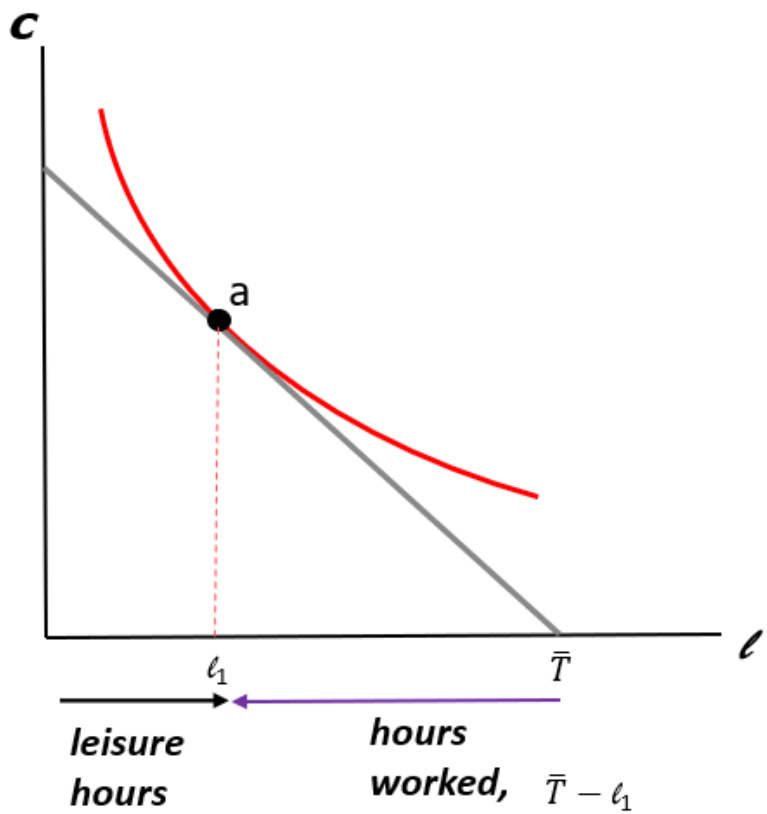
Applicaton: Consumption-Labor Supply Decisions

- Labor supply decisions of individuals comes from maximizing consumption (c) vs leisure (l):

$$\max \quad u = c \cdot l \quad s.t. \quad c = w \cdot (\bar{T} - l)$$

- Note that the slope of the constraint function is $-w$, its $x-$intercept is $\bar{T}$ and its $y-$intercept is $w \cdot \bar{T}$

Consumption-Labor Supply Decisions



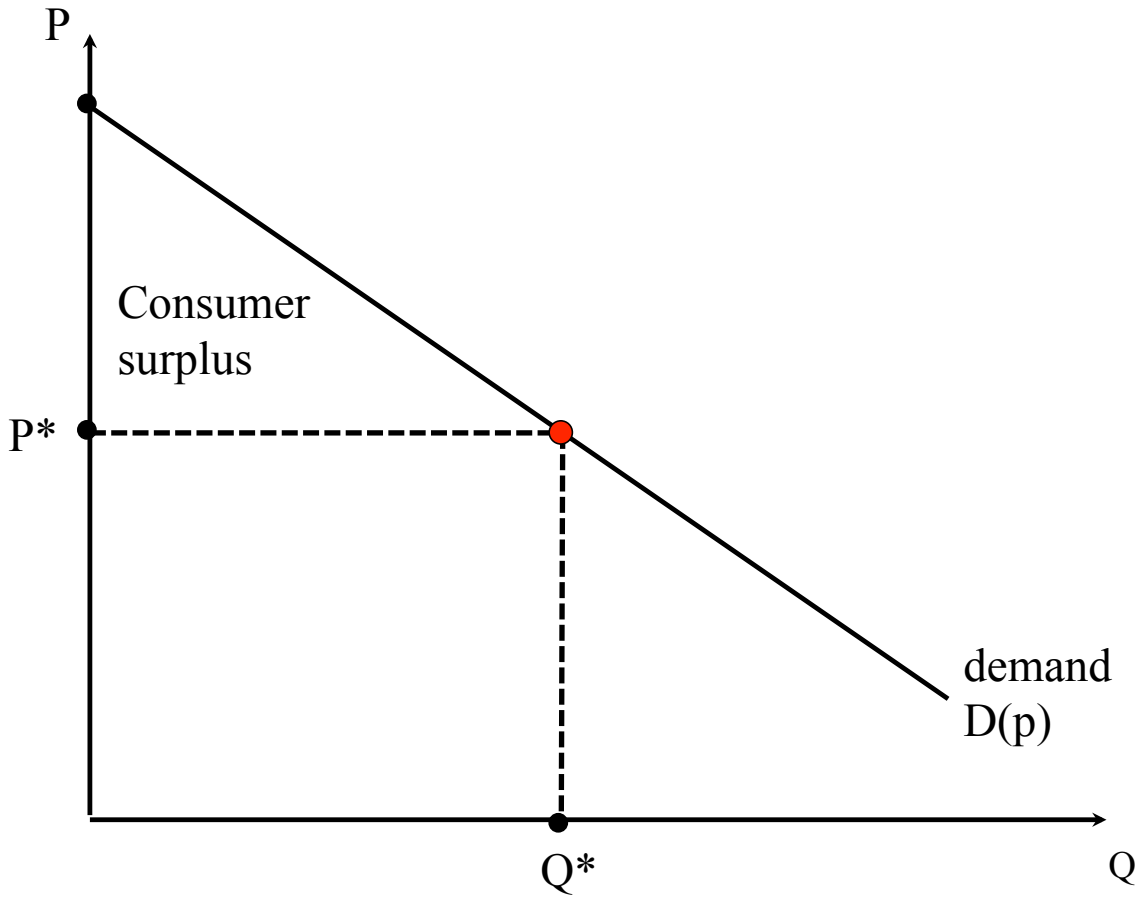
Questions:

- What happens to worked hours when wage rate increases?
- What happens to worked hours if the government gives the worker a lumpsum of \$100?

AGGREGATE DEMAND

- Each individual has a demand for each good that depends on the price p of the good.
- Aggregating across all individuals, we get aggregate demand $D(p)$ for the good
- Demand curve: quantity on X-axis, price on Y-axis
- Consumer surplus can be measured as area below the demand curve and above the price horizontal line

Consumer Surplus



ELASTICITY OF DEMAND

- The % change in demand caused by a 1% change in the price of that good:

$$\varepsilon_p = \frac{\%\text{ change in quantity demanded}}{\%\text{ change in price}} = \frac{\Delta D/D}{\Delta p/p} = \frac{dD}{dp}\frac{p}{D}$$

- Elasticities are widely used because they are **unit free**
- From the above formula, ε_p is a function of p and hence can vary with p along the demand curve
- Example: Find price elasticity of demand at $p = 1$ if demand is given by $Q_d = 100 - 2p$

PROPERTIES OF ELASTICITY OF DEMAND

1. Typically negative, since quantity demanded typically falls as price rises.
2. Typically not constant along a demand curve.
3. With vertical demand curve, demand is **perfectly inelastic** ($\varepsilon = 0$).
4. With horizontal demand curve, demand is **perfectly elastic** ($\varepsilon = -\infty$).

Other Elasticities

- We can define as many elasticity as there are relationships between any two varibles:

1. **cross-price elasticity** $= \frac{dD}{dp_2}\frac{p_2}{D}$
   - it is the ratio of the percentage change in demand for good 1 to the percentage change price of good 2
   - Typically, not zero. when positive the two are substitutes and when negative they are complements
2. **Income elasticity**: $= \frac{dD}{dy}\frac{y}{D}$
   - it is the ratio of the percentage change in demand for good 1 to the percent change in income
   - when positive the good is normal good and when negative the good is inferior good

PRODUCERS and Supply Curve

- **Supply curve**: A curve showing the quantity of a good that firms in aggregate are willing to supply at each price:
- The supply curve of a seller comes from profit maximization problem of the seller and is defined only for a firm in competitive market
- The supply curve of the firm is the marginal cost curve of the firm above the minimum average variable cost
- The market (aggregate) supply of a good is given by horizontal summation of all firms supply curve

SUPPLY CURVES

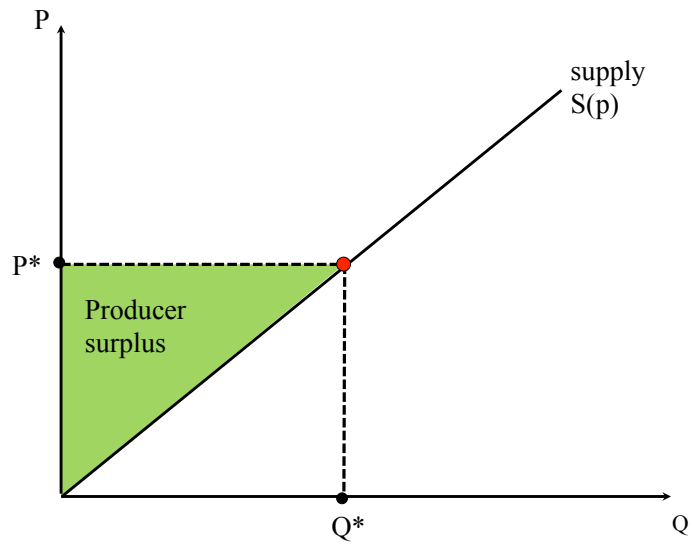
Elasticity of supply ε_S is defined as

$$\varepsilon_S = \frac{\%\text{ change in quantity supplied}}{\%\text{ change in price}} = \frac{\Delta S/S}{\Delta p/p} = \frac{dS}{dp}\frac{p}{S}$$

ε_S is a function of p and hence can vary with p along the supply curve

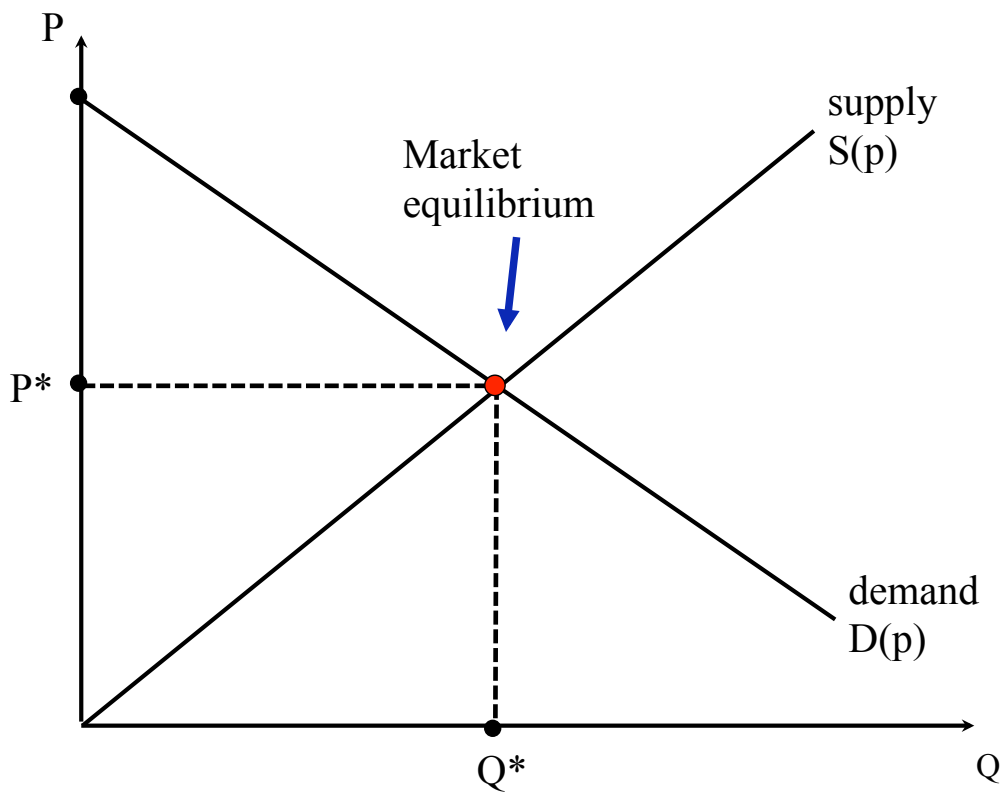
Producer Surplus

- **producer surplus** is the benefit that a seller enjoys from selling a good.
- For a competitive firm, it is given by the area below the price and above the supply curve up to the quantity sold.
- it is the same as profit (if we ignore the fixed cost) and is given



MARKET EQUILIBRIUM

- **Market equilibrium**: The equilibrium is the price p^* such that $D(p^*) = S(p^*)$
- it is a unique point and any other price implies shortage or surplus



ECONOMIC SURPLUS

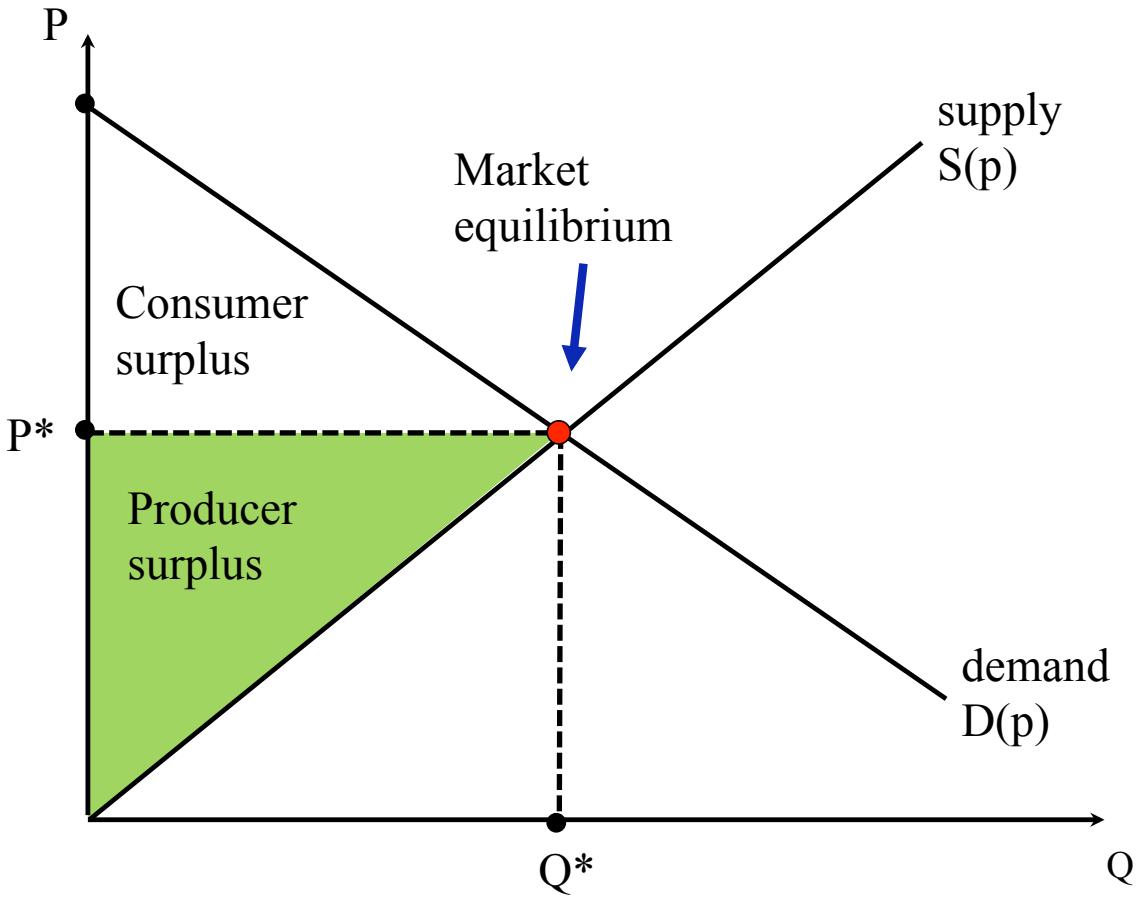
Economic surplus represents the net gains to society from all trades that are made in a particular market, and it consists of two components: consumer and producer surplus.

Consumer surplus: The benefit that consumers derive from consuming a good, above and beyond the price they paid for the good. It is the area below demand curve and above market price.

Producer surplus: The benefit producers derive from selling a good, above and beyond the cost of producing that good. It is the area above supply curve and below market price.

Total economic surplus: The sum of consumer surplus and producer surplus. It is the area above supply curve and below demand curve.

Economic or Total Surplus



Competitive Equilibrium Maximizes Economic Surplus

First Fundamental Theorem of Welfare Economics : The competitive equilibrium where supply equals demand, maximizes total economic surplus (sometimes called "efficiency' ')

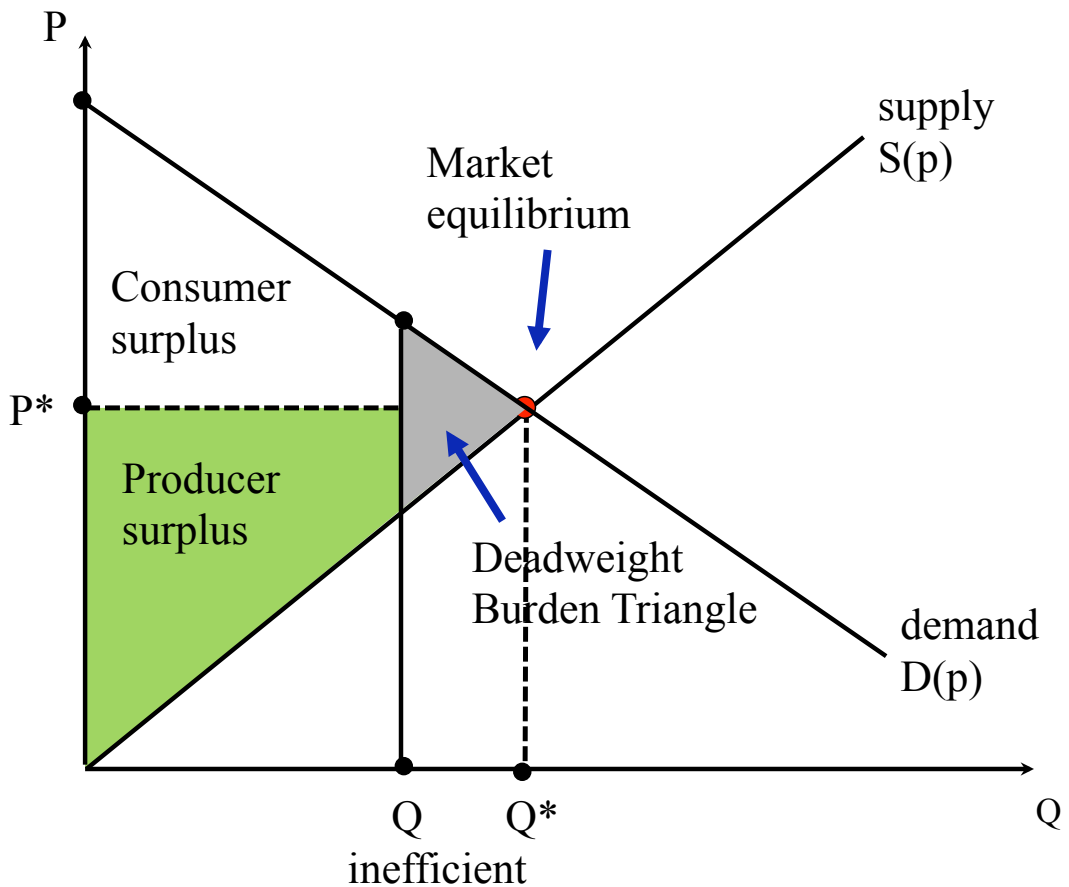
Economic surplus just counts dollars regardless of who gets them ($1 to rich producer better than $.99 to poor consumer) $\Rightarrow$ 1st welfare theorem is blind to distributional aspects

Deadweight loss: The reduction in economic surplus from denying trades for which benefits exceed costs when quantity differs from the efficient quantity

Key rule: Deadweight loss triangle points to the efficient allocation, and grows outward from there

The simple efficiency result from the 1-good diagram can be generalized into many good situtaion which is called the first welfare theorem (formulated by Arrow-Debreu, 1940s), most important result in economics

Deadweight Loss from producing less or more than equilibrium point



Generalization: 1st Welfare Theorem

- Most important result in economics (=markets work)
- **1st Welfare Theorem:** If (1) no externalities, (2) perfect competition [individuals and firms are price takers], (3) perfect information, (4) agents are rational, then private market equilibrium is **Pareto efficient**

  Pareto efficient: Impossible to find a technologically feasible allocation that improves everybody's welfare

  Pareto efficiency doesn't guarantee equitable distribution of welfare (the market can yield a Pareto efficient outcome in which a single person consuming everything)
- When there are market failures Govt intervention can potentially improve everybody's welfare

2nd Welfare Theorem

- Even with no market failures, free market outcome might generate substantial inequality. Inequality is seen as one of the biggest issue with market economies.

  2nd Welfare Theorem: Any Pareto Efficient allocation can be reached by

1. Suitable redistribution of initial endowments [individualized **lump-sum** taxes based on individual characteristics and not behavior]
2. Then letting markets work freely

   $\Rightarrow$ No conflict between efficiency and equity

2nd Welfare Theorem fallacy

- In reality, 2nd welfare theorem does not work because redistribution of initial endowments is not feasible (because initial endowments cannot be observed by the government)

  $\Rightarrow$ govt needs to use **distortionary** taxes and transfers based on economic outcomes (such as income or working situation)

  $\Rightarrow$ Conflict between efficiency and equity: **Equity-Efficiency trade-off**

  First part of class considers policies that trade-off equity and efficiency

Illustration of 2nd Welfare Theorem Fallacy

Suppose economy is populated 50% with disabled people unable to work (hence they earn \$0) and 50% with able people who can work and earn \$100

Free market outcome: disabled have \$0, able have \$100

2nd welfare theorem: govt is able to tell apart the disabled from the able [even if the able do not work]

$\Rightarrow$ can tax the able by \$50 [regardless of whether they work or not] to give \$50 to each disabled person $\Rightarrow$ the able keep working [otherwise they'd have zero income and still have to pay \$50]

Real world: govt can't tell apart disabled from non working able; hence tax must be proportional to earning

$\Rightarrow$ Now the fact that tax is proportional to degree of working reduces hours of work (May be working up to getting \$75 becomes better) $\Rightarrow$ Trade-off between equity and size of the pie

SOCIAL WELFARE FUNCTIONS

Economists incorporate distributional aspects using **social welfare functions** (instead of just adding $ of economic surplus)

Social welfare function (SWF): A function that combines the utility functions of all individuals into an overall social utility function

General idea is that one dollar to a disadvantaged person might count more than one dollar to a rich person

UTILITARIAN SOCIAL WELFARE FUNCTION

With a utilitarian social welfare function, society's goal is to maximize the sum of individual utilities:

$$SWF = U_1 + U_2 + ... + U_N$$

The utilities of all individuals are given equal weight, and summed to get total social welfare

If marginal utility of money decreases with income (satiation), utilitarian criterion values redistribution from rich to poor

Taking \$1 for a rich person decreases his utility by a small amount, giving the \$1 to a poor person increases his utility by a large amount

$\Rightarrow$ Transfers from rich to poor increase total utility

RAWLSIAN SOCIAL WELFARE FUNCTION

Rawls (1971) proposed that society's goal should be to maximize the well-being of its worst-off member. The Rawlsian SWF has the form:

$$SWF = \min(U_1, U_2, ..., U_N)$$

Since social welfare is determined by the minimum utility in society, social welfare is maximized by maximizing the well-being of the worst-off person in society (=maxi-min)

Rawlsian criterion is even more redistributive than utilitarian criterion: society wants to extract as much tax revenue as possible from the middle and rich to make transfers to the poor as large as possible

OTHER SOCIAL JUSTICE PRINCIPLES

Standard welfarist approach is based on individual utilities. This fails to capture important elements of actual debates on redistribution and fairness

1. **Just deserts**: Individuals should receive compensation congruent with their contributions (libertarian). $\Rightarrow$ Taxes should be tailored to government benefits received i.e., taxes should be based on benefits received.

2. **Commodity egalitarianism**: Society should ensure that individuals meet a set of basic needs (seen as rights) but that beyond that point income distribution is irrelevant

$\Rightarrow$ Rich countries today consider free education, universal health care, retirement/disability benefits as rights

OTHER SOCIAL JUSTICE PRINCIPLES, cont'd

3. **Equality of opportunity**: Society should ensure that all individuals have equal opportunities for success

   $\Rightarrow$ Individuals should be compensated for inequalities they are not responsible for (e.g., family background, inheritance, intrinsic ability) but not for inequalities they are responsible for (being hard working vs. loving leisure)

ACTUAL SOCIAL PREFERENCES

General conclusion: People favor redistribution if they feel inequalities are "unfair'' but views on what is fair differ

$\Rightarrow$ Redistribution supported when people don't have control [education for children, health insurance for the sick, retirement/disability benefits for the elderly/disabled unable to work]

$\Rightarrow$ Less support when people have some or full control [unemployment, being low income]

$\Rightarrow$ Less support when people don't "belong'' (us vs. them)

Conservatives tend to frame things: individuals have control (personal responsibility), govt should just enforce rules

Liberals tend to frame things: many forces in society beyond individuals' control ("we are all in this together''), society should provide nurturing

See Lakoff (1996) for how liberals and conservative think

Conclusion

- The purpose of this chapter was to introduce you to some theoretical tools that we use in public economics.
- In addition to using math, we developed some models to understand consumers and firms
- most models in economics have to do with optimizing an objective given some constraint

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